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1990 J. Phys. A: Math. Gen. 23 3169

(http://iopscience.iop.org/0305-4470/23/14/016)

J. Phys. A: Math. Gen. 23 (1990) 3169–3182. Printed in the UK

On the Legendre map in higher-order field theories

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Received 14 November 1989

Abstract. We show how the construction of a Cartan form in higher-order field theories defines a Legendre map, and how the regularity of this map may be described in terms of a sequence of maps between 'intermediate phase spaces'. We also show how semi-holonomic jets are important in this context: they allow the definition of regularity to be applied without ambiguity to the Lagrangian itself, and they permit the specification of a unique Cartan form, Legendre map and covariant phase space for second-order Lagrangians.

1. Introduction

Over the past decade there have been many studies of the global higher-order calculus of variations in several independent variables. These studies have usually been set in the context of jet bundles. Their objectives have been to construct Euler–Lagrange equations for critical sections; to obtain Cartan forms, for use in the first variation formula (the formula corresponding to the classical 'integration by parts' procedure for obtaining Euler–Lagrange equations); and to find Legendre maps, which—with a suitable definition of regularity—will permit the Euler–Lagrange equations to be recast in a first-order Hamiltonian format. The continued interest in the problem indicates that the task is by no means straightforward, particularly if the resulting constructions are supposed to reduce to the standard, well known objects in the case of a single independent variable. While a unique formulation of the Euler–Lagrange equations can be found, and a Cartan form always exists, the latter will in general not be unique if the order of the variational problem is greater than two, and if there is more than one independent variable (the case of second-order problems is rather special, and is discussed in a separate section below). As far as Legendre maps are concerned, there still seems to be no general agreement on a suitable definition.

The main approach to the global problem of finding these objects has been to provide a construction for a Cartan form, and several different techniques have been employed for this. One, used notably by Krupka (1987), has been to define a class of differential forms ('Lepagean forms') which satisfy certain properties; if a Lagrangian is equivalent (in a precise sense) to one of these Lepagean forms then the equivalence corresponds to the first variation formula, and the Lepagean form may be taken as a Cartan form. The construction of a Lepagean equivalent may be carried out locally,

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0305-4470/90/143169+14$03.50

using coordinates, and a global equivalent may be constructed by patching together the local forms using a partition of unity.

A second technique has been to reduce the problem to first order, by considering all except the highest-order derivatives as dependent variables in their own right. This has been done in several different ways. Kuperschmidt (1980) gave a global construction of a Cartan form by using repeatedly a version of this technique (see also Saunders 1987, 1989b). Aldaya and De Azcárraga (1980) considered the problem in terms of constraints and Lagrange multipliers, although there seem to be some hidden assumptions in their work. Gotay (1989) has proposed a setting for these constructions (including a Legendre map) which will form the basis of our own development later in this paper.

A third technique has been to use connections, notably in the work of García and Muñoz (1983) and of Ferraris and Francaviglia (1983). With this additional data, it is possible to define a unique Cartan form. The works cited use pairs of connections, although Kolář (1984), in a paper employing yet another method of constructing a Cartan form, has shown that a single connection is adequate for making a unique choice of such a form. Similar to the use of connections has been an approach (de León and Rodrigues 1989) where the independent variables are assumed to be taken from $\mathbb{R}^m$, rather than from a more general manifold: use of the standard coordinate system in $\mathbb{R}^m$ then amounts to the specification of a connection.

The purpose of this paper is to show how the construction of a Cartan form by the technique of reduction to first order may be used to give definitions both of a Legendre map, and of regularity. Since the definitions reduce to the standard ones in the case of a single independent variable, we hope that they will be accepted in the present context. Our new results arise from the use of semi-holonomic jets, which are derived from the Spencer cohomology of jet bundles (Modugno and Mangiarotti 1983, Pommaret 1984). These are used to make a unique choice of Cartan form in second-order theories, and to provide an unambiguous definition of regularity: they correspond to the quasisymmetric operators of Kolář (1984). We also give a geometric decomposition of the Legendre map as a sequence of maps, following a technique proposed by Gràcia *et al* (1989) for the one-dimensional case. This decomposition allows us to express our regularity condition as a set of several conditions, and in coordinates these just correspond to the conditions given by Shadwick (1982).

The layout of this paper is as follows. In section 2 we describe some constructions involving affine bundles and jet bundles: wherever notation is not described explicitly, we follow that of Saunders (1989a). In section 3 we review the first-order theory, following Cariñena *et al* (1989) and in section 4 we introduce the higher-order theory. Section 5 contains our definition of regularity, and in section 6 we describe some special aspects of the second-order case.

2. Affine bundles and jet bundles

Let A be an n-dimensional affine space, modelled on the vector space V. Since the real numbers $\mathbb{R}$ form an affine space the set $A^\dagger$ of real-valued affine functions on A may be given the structure of an affine space of dimension $(n+1)$. If $\mathbb{R}$ is regarded as a vector space then $A^\dagger$ has a distinguished element (the zero function), so it may also be regarded as a vector space. Some authors call $A^\dagger$ the *dual* of A; we shall, however, prefer to call $A^\dagger$ the *extended dual* of A, and reserve the term 'dual' for the n-dimensional affine space $A^* = A^\dagger / A^c$, where A^c is the affine subspace of $A^\dagger$ consisting of the constant functions. Since A^* also has a distinguished element, namely the equivalence class of constant functions, it too may be regarded as a vector space, and is then canonically isomorphic to the vector space V^* dual to V. Note that, in general, A^c does not have a distinguished complement in $A^\dagger$.

Similar constructions to those just described may be applied fibrewise in the context of affine bundles, and we shall be concerned in particular with those jet bundles which have an affine structure. If (E, π, M) is a bundle (with total space E, orientable base space M and projection π) then its kth jet bundle will be written as $(J^k\pi, \pi_k, M)$. For each k the bundle $(J^k\pi, \pi_{k,k-1}, J^{k-1}\pi)$ will be an affine bundle. Furthermore, on taking repeated jets, we find that $(J^1\pi_{k-1}, (\pi_{k-1})_{1,0}, J^{k-1}\pi)$ is also an affine bundle, and that the canonical injection $\iota_{1,k-1} : J^k\pi \longrightarrow J^1\pi_{k-1}$ defines a morphism of affine bundles over $J^{k-1}\pi$. We remark for later use that the vector bundle corresponding to $(J^1\pi_{k-1}, (\pi_{k-1})_{1,0}, J^{k-1}\pi)$ has total space $\pi_{k-1}^* T^*M \otimes V\pi_{k-1}$, where $V\pi_{k-1}$ is the sub-bundle of $TJ^{k-1}\pi$ containing vectors vertical over M. We shall also consider the manifold $\widehat{J^k\pi}$, which is defined to be the submanifold of $J^1\pi_{k-1}$ where the two maps $\iota_{1,k-2} \circ (\pi_{k-1})_{1,0}$ and $j^1\pi_{k-1,k-2}$ from $J^1\pi_{k-1}$ to $J^1\pi_{k-2}$ are equal. This, too, is fibred over $J^{k-1}\pi$, and defines a third affine bundle $(\widehat{J^k\pi}, (\pi_{k-1})_{1,0}, J^{k-1}\pi)$. We then have the relations

$$\iota_{1,k-1}(J^k\pi) \subset \widehat{J^k\pi} \subset J^1\pi_{k-1}$$

and the three manifolds are said to contain holonomic, semi-holonomic and non-holonomic jets respectively.

We shall normally take a fixed volume form Ω on the base space M, and choose coordinates x^i such that Ω may be written as $\mathrm{d}^m x = \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}^m x$; we shall also write $\mathrm{d}^{m-1}x_i$ for the contraction $\partial/\partial x^i \lrcorner \, \mathrm{d}^m x$. If adapted coordinates (x^i, u^α) are chosen on the total space E, then the induced coordinates on $J^k\pi$ are (x^i, u^α_I), where I is a multi-index satisfying $0 \leq |I| \leq k$. The induced coordinates on $J^1\pi_{k-1}$ are $(x^i, u^\alpha_I, u^\alpha_{I,i})$ where now I satisfies $0 \leq |I| \leq k-1$. The submanifold $\widehat{J^k\pi}$ is defined locally by the equations

$$u^\alpha_{I,i} = u^\alpha_{I+1_i}$$

for $0 \leq |I| \leq k-2$; the submanifold $\iota_{1,k-1}(J^k\pi)$ is then defined by the further equations

$$u^\alpha_{I,i} = u^\alpha_{J,j}$$

whenever $|I| = |J| = k-1$ and $I + 1_i = J + 1_j$. In coordinate representations, we shall generally use the same symbol to represent a function or form, and its pull-back to a higher-order jet manifold.

It is important to note that, in general, $\iota_{1,k-1}(J^k\pi)$ does not have a distinguished complement in $J^1\pi_{k-1}$. In contrast, however, $\iota_{1,k-1}(J^k\pi)$ *does* have a distinguished complement in the semi-holonomic manifold $\widehat{J^k\pi}$. To describe this complement explicitly, let $\delta : \bigwedge^r T^*M \otimes S^q T^*M \longrightarrow \bigwedge^{r+1} T^*M \otimes S^{q-1} T^*M$ denote the Spencer coboundary operator. This is the vector bundle morphism induced by the inclusion map $S^q T^*M \longrightarrow T^*M \otimes S^{q-1} T^*M$. At each point $p \in M$ this morphism may be written in coordinates as

$$\delta(\mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_r} \otimes \mathrm{d}x^I)_p = \sum_{J+1_j=I} I(j)(\mathrm{d}x^j \wedge \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_r} \otimes \mathrm{d}x^J)_p$$

where $I(j)$ represents the number of occurrences of the index j in the multi-index I; the sequence formed by successive maps δ is exact. We shall be interested in the particular map $\delta : T^*M \otimes S^{k-1}T^*M \longrightarrow \bigwedge^2 T^*M \otimes S^{k-2}T^*M$; if we consider the pull-backs of the domain and codomain bundles to $J^{k-1}\pi$, and take the tensor product of the induced map with the identity on $\pi^*_{k-1,0}(V\pi)$, we obtain a map

$$\delta : \pi^*_{k-1}(T^*M \otimes S^{k-1}T^*M) \otimes \pi^*_{k-1,0}V\pi \longrightarrow \pi^*_{k-1}(\textstyle\bigwedge^2 T^*M \otimes S^{k-2}T^*M) \otimes \pi^*_{k-1,0}V\pi.$$

It may be shown (Pommaret 1984) that

$$\widehat{J^k\pi} \cong J^k\pi \times_{J^{k-1}\pi} \operatorname{Im}(\delta)$$

and we shall let $\tau_k : \widehat{J^k\pi} \longrightarrow J^k\pi$ be the corresponding projection. Note in particular that, when $k = 2$, the exactness of the Spencer sequence implies that $\operatorname{Im}(\delta)$ is the whole of $\pi^*_1 \bigwedge^2 T^*M \otimes \pi^*_{1,0}V\pi$.

3. The first-order theory

Let $L : J^1\pi \longrightarrow \mathbb{R}$ be a Lagrangian. The Cartan form of L may be defined to be the m-form Θ_L on $J^1\pi$ given by the equation $\Theta_L = S_\Omega \lrcorner\, \mathrm{d}L + L\pi^*_1\Omega$, where S_Ω is the vertical vector-valued m-form on $J^1\pi$ corresponding to a given volume form Ω on M. In coordinates

$$\Theta_L = \frac{\partial L}{\partial u^\alpha_i}(\mathrm{d}u^\alpha - u^\alpha_j \mathrm{d}x^j) \wedge \mathrm{d}^{m-1}x_i + L\,\mathrm{d}^m x.$$

The Cartan form satisfies the first variation formula

$$\mathrm{E}(\mathrm{d}L) = \pi^*_{2,1}\mathrm{d}L \wedge \pi^*_2\Omega + \mathrm{d}_h\Theta_L$$

where $\mathrm{E}(\mathrm{d}L)$ denotes the Euler–Lagrange form

$$\mathrm{E}(\mathrm{d}L) = \left(\frac{\partial L}{\partial u^\alpha} - \frac{\mathrm{d}}{\mathrm{d}x^i}\frac{\partial L}{\partial u^\alpha_i}\right)\mathrm{d}u^\alpha \wedge \mathrm{d}^m x.$$

The set of points in $J^2\pi$ where $\mathrm{E}(\mathrm{d}L)$ vanishes is called the Euler–Lagrange manifold for L, and if ϕ is any critical section for L then $j^2\phi$ must take its values in this manifold.

In order to construct a Legendre map, we may use the theory of affine duals as in (Cariñena *et al* 1989). This theory involves the affine bundle $(J^1\pi, \pi_{1,0}, E)$, its extended dual bundle $(J^1\pi^\dagger, \pi^\dagger_{1,0}, E)$, and its dual bundle $(J^1\pi^*, \pi^*_{1,0}, E)$. We shall write $\mu : J^1\pi^\dagger \longrightarrow J^1\pi^*$ for the canonical projection from the extended dual to the dual, so that $\pi^*_{1,0} \circ \mu = \pi^\dagger_{1,0}$. We shall let $(x^i, u^\alpha, p, p^i_\alpha)$ be the induced coordinates on $J^1\pi^\dagger$, and $(x^i, u^\alpha, p^i_\alpha)$ be the corresponding coordinates on $J^1\pi^*$.

For each point $a \in E$, the fibre $(J^1\pi)_a$ is an affine space, and we shall denote the restriction of L to this fibre by L_a. If $j^1_p\phi$ is any point of $J^1\pi$, the differential $\mathrm{d}L_{\phi(p)}$ may then be evaluated at $j^1_p\phi$ to give a real-valued affine map on the fibre $(J^1\pi)_{\phi(p)}$.

It follows that this map $\mathrm{d}L_{\phi(p)}\Big|_{j^1_p\phi}$ is an element of the extended dual space $(J^1\pi)^{\dagger}_{\phi(p)}$, and we shall denote the correspondence $J^1\pi \longrightarrow J^1\pi^{\dagger}$, $j^1_p\phi \longmapsto \mathrm{d}L_{\phi(p)}\Big|_{j^1_p\phi}$ by Leg_L; the composition $\mu \circ \mathrm{Leg}_L$ will be denoted leg_L. (In Cariñena *et al* 1989, these two maps were denoted by $\widetilde{\mathcal{F}L}$ and $\mathcal{F}L$ respectively.) In coordinates, we have for Leg_L

$$p \circ \mathrm{Leg}_L = L - u^\alpha_i \frac{\partial L}{\partial u^\alpha_i}$$

$$p^i_\alpha \circ \mathrm{Leg}_L = \frac{\partial L}{\partial u^\alpha_i}$$

and for leg_L

$$p^i_\alpha \circ \mathrm{leg}_L = \frac{\partial L}{\partial u^\alpha_i}.$$

We may use these relationships to define local momentum functions on $J^1\pi$: we shall set $P^i_\alpha = p^i_\alpha \circ \mathrm{Leg}_L$ and $P = p \circ \mathrm{Leg}_L$, so that we may rewrite the coordinate description of the Cartan form as

$$\Theta_L = P^i_\alpha \mathrm{d}u^\alpha \wedge \mathrm{d}^{m-1}x_i + P\,\mathrm{d}^m x.$$

The reason for adopting this terminology is that each of the maps Leg_L and leg_L has some claim to be called the Legendre map for L. On the one hand, the manifolds $J^1\pi$ and $J^1\pi^*$ have the same dimension, and the usual condition for regularity in first-order field theories, $\det\left(\partial^2 L/\partial u^\alpha_i\,\partial u^\beta_j\right) \neq 0$, corresponds to the map leg_L being a local diffeomorphism; we may also say that the Lagrangian is hyper-regular if leg_L is a global diffeomorphism. In these circumstances, it is reasonable to call leg_L a Legendre transformation. We may define a Hamiltonian system on $J^1\pi^*$ as a section h of the bundle $(J^1\pi^{\dagger}, \mu, J^1\pi^*)$, so that $H = p \circ h$ is a local Hamiltonian function on $J^1\pi^*$, and then a hyper-regular Lagrangian induces a bijection between solutions of the Euler–Lagrange equations, and solutions of Hamilton's equations.

On the other hand, the map Leg_L has a close connection with the Cartan form Θ_L. This arises because the volume form Ω defines a natural identification of the dual bundle $J^1\pi^*$ with a bundle of m-covectors over E, namely the sub-bundle of $\bigwedge^m T^*E$ containing those m-covectors θ satisfying $i_\xi i_\eta \theta = 0$ whenever both the vectors $\xi, \eta \in TE$ are vertical over M. In coordinates, such an element θ would be written as

$$\theta = \theta^i_\alpha \mathrm{d}u^\alpha \wedge \mathrm{d}^{m-1}x_i + \theta_0 \mathrm{d}^m x.$$

The correspondence between this bundle of m-covectors and $J^1\pi^{\dagger}$ is then given in coordinates by letting $\theta(j^1_p\phi) = \theta^i_\alpha u^\alpha_i(j^1_p\phi) + \theta_0$ whenever θ and $j^1_p\phi$ project to the same point $\phi(p)$ of E. This identification of $J^1\pi^{\dagger}$ as a bundle of m-covectors means that it carries a canonical m-form Θ, whose value at $\theta \in J^1\pi^{\dagger}$ is just the pull-back by $\pi^{\dagger}_{1,0}$ of θ from E to $J^1\pi^{\dagger}$: in coordinates,

$$\Theta = p^i_\alpha \mathrm{d}u^\alpha \wedge \mathrm{d}^{m-1}x_i + p\,\mathrm{d}^m x.$$

It is clear from this coordinate description that the Legendre map and the Cartan form are related by the formula $\Theta_L = \mathrm{Leg}^*_L \Theta$. Indeed, since Θ_L is semi-basic over E, it may be considered as an 'm-form along $\pi_{1,0}$': that is, a map assigning m-covectors on E to points in $J^1\pi$—in other words a map $J^1\pi \longrightarrow J^1\pi^{\dagger}$. With this interpretation, the Cartan form Θ_L and the Legendre map Leg_L are not merely closely related: they are, in fact, identical.

4. The higher-order theory

Let $L : J^k\pi \longrightarrow \mathbb{R}$ be a kth-order Lagrangian. By analogy with the first-order theory, one objective of the higher-order theory is to construct a Cartan form and an Euler–Lagrange form so that the first variation formula

$$\mathrm{E}(\mathrm{d}L) = \pi^*_{2k,k}\mathrm{d}L \wedge \pi^*_{2k}\Omega + \mathrm{d}_h\Theta_L$$

will hold. Most of the published work on this problem agrees that there is a unique such Euler–Lagrange form, whereas (in the absence of additional data) there is normally a lack of uniqueness in the Cartan form. In section 6 we shall describe an explicit construction of this formula in the second-order case; for the moment, we shall simply repeat the standard results. The coordinate representation of the Euler–Lagrange form is

$$\mathrm{E}(\mathrm{d}L) = \sum_{|I|=0}^{k} (-1)^{|I|} \frac{\mathrm{d}^{|I|}}{\mathrm{d}x^I} \frac{\partial L}{\partial u^\alpha_I}\, \mathrm{d}u^\alpha \wedge \mathrm{d}^m x$$

and the representation of any Cartan form is

$$\Theta_L = \sum_{|I|=0}^{k-1} P^{I,i}_\alpha \mathrm{d}u^\alpha_I \wedge \mathrm{d}^{m-1}x_i + P\,\mathrm{d}^m x$$

where

$$P = L\,\mathrm{d}^m x - \sum_{|I|=0}^{k-1} u^\alpha_{I+1_i} P^{I,i}_\alpha$$

where the local momentum functions $P^{I,i}_\alpha$ satisfy

$$P^{I,i}_\alpha = \frac{I(i)+1}{|I|+1}\left(\frac{\partial L}{\partial u^\alpha_{I+1_i}} + Q^{I,i}_\alpha\right)$$

for $|I| = k-1$ and

$$P^{I,i}_\alpha = \frac{I(i)+1}{|I|+1}\left(\frac{\partial L}{\partial u^\alpha_{I+1_i}} + Q^{I,i}_\alpha - \frac{\mathrm{d}}{\mathrm{d}x^j} P^{I+1_i,j}_\alpha\right)$$

for $0 \le |I| < k-1$, and where the functions $Q^{I,i}_\alpha$ satisfy $\sum_{I+1_i=J} Q^{I,i}_\alpha = 0$ for each multi-index J. The functions $Q^{I,i}_\alpha$ are in general not determined by the Lagrangian L, and give rise to the lack of uniqeness of the Cartan form.

The main constructions in this paper involve a Legendre map corresponding to a Cartan form Θ_L, and we shall define such a map by analogy with the first-order theory. Since Θ_L is defined on $J^{2k-1}\pi$ and is semi-basic over $J^{k-1}\pi$, we may regard it as a map from $J^{2k-1}\pi$ to $J^1\pi^\dagger_{k-1}$ fibred over $J^{k-1}\pi$, and we shall denote this map by Leg_L; this will be the map which we shall call the Legendre map for L. In a similar way, we shall denote the composite of Leg_L with the projection $J^1\pi^\dagger_{k-1} \longrightarrow J^1\pi^*_{k-1}$

by leg_L. The lack of uniqueness of the Cartan form will obviously carry over into a lack of uniqueness of the Legendre map.

Although we shall not repeat the explicit construction of a Cartan form for L, it is nevertheless important for us to explain the first stage of its construction, as we can describe a technique whereby those non-symmetric quantities $Q_\alpha^{I,i}$ with $|I| = k-1$ may be chosen to be zero. We shall use the method of reduction to first order, and assume that there is a first-order Lagrangian $\bar{L} : J^1\pi_{k-1} \longrightarrow \mathbb{R}$ satisfying $\iota_{1,k-1}^*\bar{L} = L$. We may then apply the first variation formula for $\bar{L}$, giving

$$\mathrm{E}(\mathrm{d}\bar{L}) = (\pi_{k-1})_{2,1}^*\mathrm{d}\bar{L} \wedge (\pi_{k-1})_2^*\Omega + \mathrm{d}_h\Theta_{\bar{L}}$$

as an $(m+1)$-form on $J^2\pi_{k-1}$. Now the pull-back $\iota_{2,k-1}^*\mathrm{E}(\mathrm{d}\bar{L})$ is semi-basic over $J^{k-1}\pi$, and so a version of the first variation formula may be applied to continue this process, finishing with a form semi-basic over the total space E; the Cartan form for L is then the sum of the various 'partial Cartan forms' constructed using this procedure. The point to note is that the non-uniqueness in the Cartan form arises from the need to extend forms defined on holonomic manifolds $J^{k+r-1}\pi$ which are semi-basic over $J^{k-r+1}\pi$ (for $1 \le r \le k$) to forms defined on non-holonomic manifolds $J^{2r-1}\pi_{k-r}$ which are semi-basic over $J^1\pi_{k-r}$, and that the only occasion where there may be non-symmetric terms added to the highest-order derivatives $\partial L/\partial u_I^\alpha$ (with $|I| = k$) occurs in the case where $r = 1$: that is, where $\mathrm{d}L$ is extended to $\mathrm{d}\bar{L}$. We may now take advantage of the existence of a complement to $J^k\pi$ in $\widehat{J^k\pi}$, and specify that any extension $\bar{L}$ should satisfy $\bar{L}\big|_{\widehat{J^k\pi}} = \tau_k^*L$, so that its values on the semi-holonomic manifold $\widehat{J^k\pi}$ are determined. From the coordinate relationship

$$u_{I,i}^\alpha = \frac{k}{I(i)+1}\tau_k^* u_{I+1_i}^\alpha$$

with $|I| = k-1$ we find that

$$\iota_{1,k-1}^*\frac{\partial\bar{L}}{\partial u_{I,i}^\alpha} = \frac{I(i)+1}{k}\frac{\partial L}{\partial u_{I+1_i}^\alpha}.$$

The coordinate expression of $\Theta_{\bar{L}}$ therefore becomes

$$\Theta_{\bar{L}} = \sum_{|I|=k-1}\frac{I(i)+1}{k}\frac{\partial L}{\partial u_{I+1_i}^\alpha}\mathrm{d}u_I^\alpha \wedge \mathrm{d}^{m-1}x_i + \ldots$$

so that $Q_\alpha^{I,i} = 0$ for $|I| = k-1$: in other words, the highest-order momentum functions $P_\alpha^{I,i}$ are totally symmetric.

5. Regularity

In first-order theories, a Lagrangian is called *regular* if the corresponding Legendre map $\mathrm{leg}_L : J^1\pi \longrightarrow J^1\pi^*$ is a local diffeomorphism. A similar definition is used in higher-order theories with a one-dimensional base manifold. The complication when $\dim M > 1$ and $k > 1$ is that the map $\mathrm{leg}_L : J^{2k-1}\pi \longrightarrow J^1\pi_{k-1}^*$ can be neither locally injective nor locally surjective. It is, nevertheless, possible to define the regularity of a higher-order Lagrangian L in terms of the rank of its Legendre map, and we shall see that this is independent of the choice of extension $\bar{L}$.

Our definition will be that the Lagrangian L is regular if, at each point of $J^{2k-1}\pi$, the rank of the Legendre map leg_L equals $\dim J^k\pi + \dim J^{k-1}\pi - \dim E$: we shall denote this latter number by $N(E, M, k)$. Note that, when $k = 1$, $N(E, M, k)$ equals both $\dim J^1\pi$ and $\dim J^1\pi^*$, and that when $\dim M = 1$ it equals both $\dim J^{2k-1}\pi$ and $\dim J^1\pi^*_{k-1}$: in both these cases, therefore, our definition reduces to the standard one, and then $N(E, M, k)$ is actually the dimension of the phase space. It turns out that for second-order field theories it is also possible to define a canonical phase space whose dimension is $N(E, M, 2)$, as we shall show in section 6. In general, however, it is not possible to express $N(E, M, k)$ as the dimension of a canonically defined covariant phase space, because the definition of such a space depends on the choice of Legendre map.

We have been led to this definition of regularity by studying the generalisation, to $\dim M > 1$, of a construction introduced by Gràcia *et al* (1989) in the one-dimensional case. To demonstrate how our definition arises, we shall suppose that a Lagrangian L is given, and that a Cartan form Θ_L and associated Legendre map $\mathrm{Leg}_L : J^{2k-1}\pi \longrightarrow J^1\pi^\dagger_{k-1}$ have been chosen. We shall then decompose the map Leg_L into a sequence of maps $L_s : K_s \longrightarrow K_{s+1}$ for $0 \leq s \leq k-1$, where each K_s is a submanifold of $J^{2k-1-s}\pi \times_{J^{k-1}\pi} J^1\pi^\dagger_{k-1}$, and where for each s the map L_s introduces the local momentum functions $P^{I,i}_\alpha$ where $|I| = s$. The first step in this decomposition will be to give a definition of K_s, and for this we shall generalise the technique used by Gràcia *et al* (1989). We shall need to use certain vector-valued 1-forms S_ω on the jet manifold $J^{k-1}\pi$ which generalise the almost-tangent structures on higher-order tangent manifolds: these may be defined intrinsically (Saunders 1987) and in coordinates are written as

$$S_\omega = \sum_{|J+K|=0}^{k-2} \frac{(J+K+1_i)!}{(J+1_i)!\,K!} \frac{\partial^{|J|}\omega_i}{\partial x^J} (\mathrm{d}u^\alpha_K - u^\alpha_{K+1_j}\mathrm{d}x^j) \otimes \frac{\partial}{\partial u^\alpha_{J+K+1_i}}$$

where $\omega = \omega_j \mathrm{d}x^j$ is a closed 1-form on the base manifold M. We shall use the operators S_ω to construct two families of m-forms on $J^{2k-1-s}\pi \times_{J^{k-1}\pi} J^1\pi^\dagger_{k-1}$, where each family is parametrised by an s-tuple $(\omega^1, \ldots, \omega^s)$ of closed 1-forms on M; we shall let K_s be the submanifold where, for every such s-tuple, the corresponding two m-forms are equal. The construction involves the repeated contraction of vector-valued 1-forms S_ω with two given m-forms; although the m-forms to which we shall apply the operators S_ω are defined on $J^{2k-1}\pi$ and $J^1\pi^\dagger_{k-1}$ respectively, they are both semi-basic over $J^{k-1}\pi$, and so the pointwise operation of contraction may be considered to be taking place on $J^{k-1}\pi$.

The first family of m-forms will be constructed from the canonical m-form Θ on $J^1\pi^\dagger_{k-1}$. The contraction $S_{\omega^1} \lrcorner \ldots \lrcorner S_{\omega^s} \lrcorner \Theta$ is another m-form on $J^1\pi^\dagger_{k-1}$, and this may be pulled back by the fibre product projection to an m-form $\Theta[\omega^1, \ldots, \omega^s]$ on $J^{2k-1-s}\pi \times_{J^{k-1}\pi} J^1\pi^\dagger_{k-1}$.

The second family of m-forms will be constructed from the Cartan form Θ_L on $J^{2k-1}\pi$. The contraction $S_{\omega^1} \lrcorner \ldots \lrcorner S_{\omega^s} \lrcorner \Theta_L$ is another m-form on $J^{2k-1}\pi$, and it is semi-basic over $J^{2k-1-s}\pi$ because Θ_L is semi-basic over $J^{k-1}\pi$ and the effect of each contraction with S_{ω^i} is to reduce the order of each derivative differential $\mathrm{d}u^\alpha_I$ by at least one. Indeed, this m-form is actually basic over $J^{2k-1-s}\pi$, because the coefficients which involve the s highest-order derivative coordinates are attached to the s lowest-order derivative differentials, and these vanish after contraction with $S_{\omega^1} \lrcorner \ldots \lrcorner S_{\omega^s}$ (the coefficients in the representation of S_{ω^i} are of course pulled back from M). It follows that the m-form $S_{\omega^1} \lrcorner \ldots \lrcorner S_{\omega^s} \lrcorner \Theta_L$ is projectable onto $J^{2k-1-s}\pi$, and the result may be pulled back by the fibre product projection to an m-form $\Theta_L[\omega^1, \ldots, \omega^s]$ on $J^{2k-1-s}\pi \times_{J^{k-1}\pi} J^1\pi^\dagger_{k-1}$.

Now define K_s to be the subset of points a where $\Theta[\omega^1, \ldots, \omega^s]_a = \Theta_L[\omega^1, \ldots, \omega^s]_a$ for every s-tuple $(\omega^1, \ldots, \omega^s)$. We claim that, for $1 \leq s \leq k-1$, K_s is just the submanifold given in coordinates by $p^{I,i}_\alpha = P^{I,i}_\alpha$ for $s \leq |I| \leq k-1$, where $p^{I,i}_\alpha$ are the coordinate functions pulled back from $J^1\pi^\dagger_{k-1}$, and $P^{I,i}_\alpha$ are the pull-backs of the local momentum functions described in section 4. To see this, note that

$$\Theta = \sum_{|I|=0}^{k-1} p^{I,i}_\alpha \mathrm{d}u^\alpha_I \wedge \mathrm{d}^{m-1}x_i + p\,\mathrm{d}^m x$$

so that

$$\begin{aligned}\Theta[\omega^1, \ldots, \omega^s] = &\sum_{|I|=s}^{k-1} \sum_{(*)} p^{I,i}_\alpha \omega^1_{j_1} \ldots \omega^s_{j_s} (\mathrm{d}u^\alpha_J - u^\alpha_{J+1_k} \mathrm{d}x^k) \wedge \mathrm{d}^{m-1}x_i \\ &+ \sum_{|I|=s+1}^{k-1} \sum_{(\dagger)} p^{I,i}_\alpha F^\dagger(\omega^1, \ldots, \omega^s)(\mathrm{d}u^\alpha_K - u^\alpha_{K+1_k} \mathrm{d}x^k) \wedge \mathrm{d}^{m-1}x_i\end{aligned}$$

where the sum indicated $(*)$ is over all J such that $J + 1_{j_1} + \ldots + 1_{j_s} = I$ (so that $|J| \leq k-1-s$), where the sum indicated $(\dagger)$ involves the multi-index K satisfying $|K| < k-1-s$, and where the functions $F^\dagger$ involve the coefficients of the ω^a and their derivatives. On the other hand

$$\Theta_L = \sum_{|I|=0}^{k-1} P^{I,i}_\alpha \mathrm{d}u^\alpha_I \wedge \mathrm{d}^{m-1}x_i + P\,\mathrm{d}^m x$$

so that

$$\begin{aligned}\Theta_L[\omega^1, \ldots, \omega^s] = &\sum_{|I|=s}^{k-1} \sum_{(*)} P^{I,i}_\alpha \omega^1_{j_1} \ldots \omega^s_{j_s} (\mathrm{d}u^\alpha_J - u^\alpha_{J+1_k} \mathrm{d}x^k) \wedge \mathrm{d}^{m-1}x_i \\ &+ \sum_{|I|=s+1}^{k-1} \sum_{(\dagger)} P^{I,i}_\alpha F^\dagger(\omega^1, \ldots, \omega^s)(\mathrm{d}u^\alpha_K - u^\alpha_{K+1_k} \mathrm{d}x^k) \wedge \mathrm{d}^{m-1}x_i\end{aligned}$$

where $(*)$, $(\dagger)$ and $F^\dagger$ have exactly the same meaning as before. If each ω^a is taken to be locally equal to one of the coordinate functions $\mathrm{d}x^i$ then all the functions $F^\dagger$ vanish; by considering all possible combinations of the $\mathrm{d}x^i$ we see that $p^{I,i}_\alpha = P^{I,i}_\alpha$ for $s \leq |I| \leq k-1$ at every point in K_s. It is then clear that considering any other s-tuple $(\omega^1, \ldots, \omega^s)$ introduces no further restrictions because the functions $F^\dagger$ and the sum $(\dagger)$ are the same for both the m-forms, and only involve those $p^{I,i}_\alpha$ and $P^{I,i}_\alpha$ which are already known to be equal.

We still have to consider the cases $s = 0$ and $s = k$. When $s = 0$ it is clear that K_0 is given by $p_\alpha^{I,i} = P_\alpha^{I,i}$ for $0 \le |I| \le k-1$, together with the further restriction $p = P$, and this submanifold is just the graph of Leg_L in $J^{2k-1}\pi \times_{J^{k-1}\pi} J^1\pi_{k-1}^\dagger$, and so is diffeomorphic to $J^{2k-1}\pi$. When $s = k$ both the m-forms vanish, so that K_k is the whole of $J^{k-1}\pi \times_{J^{k-1}\pi} J^1\pi_{k-1}^\dagger$, which of course is diffeomorphic to $J^1\pi_{k-1}^\dagger$.

We shall now define the map $L_s : K_s \longrightarrow K_{s+1}$ to be the restriction to K_s of the canonical map

$$\pi_{2k-1-s,2k-2-s} \times \mathrm{id} : J^{2k-1-s}\pi \times_{J^{k-1}\pi} J^1\pi_{k-1}^\dagger \longrightarrow J^{2k-2-s}\pi \times_{J^{k-1}\pi} J^1\pi_{k-1}^\dagger .$$

Of course we must check that this restriction does actually take its values in K_{s+1}, but this is clear because $\pi_{2k-1-s,2k-2-s}$ simply forgets the derivative coordinates of order $2k-1-s$, and these coordinates only appear in the momentum functions $P_\alpha^{I,i}$ with $|I| \le s$ and so are not involved in the constraint equations for K_{s+1}. We therefore have the sequence of maps

$$J^{2k-1}\pi \cong K_0 \xrightarrow{L_0} K_1 \xrightarrow{L_1} \cdots \xrightarrow{L_{k-1}} K_k \cong J^1\pi_{k-1}^\dagger$$

and it is evident from the coordinate representation that

$$L_{k-1} \circ \ldots \circ L_0 = \mathrm{Leg}_L .$$

Finally we observe that this sequence of submanifolds and maps is projectable in its entirety from $J^{2k-1-s}\pi \times_{J^{k-1}\pi} J^1\pi_{k-1}^\dagger$ to $J^{2k-1-s}\pi \times_{J^{k-1}\pi} J^1\pi_{k-1}^*$: this is because none of the submanifolds K_s $(1 \le s \le k)$ and none of the maps L_s $(0 \le s \le k-1)$ involves the coordinate p on $J^1\pi_{k-1}^\dagger$. We shall let the projected sequence be

$$J^{2k-1}\pi \cong \bar{K}_0 \xrightarrow{l_0} \bar{K}_1 \xrightarrow{l_1} \cdots \xrightarrow{l_{k-1}} \bar{K}_k \cong J^1\pi_{k-1}^*$$

and now

$$l_{k-1} \circ \ldots \circ l_0 = \mathrm{leg}_L .$$

If we take coordinates $(x^i, u_J^\beta, p_\alpha^{I,i})$ on $\bar{K}_s$, with $0 \le |J| \le 2k-1-s$ and $0 \le |I| \le s-1$, then

$$\begin{aligned} x^i \circ l_s &= x^i \\ u_J^\beta \circ l_s &= u_J^\beta \qquad && 0 \le |J| \le 2k-2-s \\ p_\alpha^{I,i} \circ l_s &= p_\alpha^{I,i} \qquad && 0 \le |I| \le s-1 \end{aligned}$$

the only non-trivial effect of l_s may be seen in the functions $P_\alpha^{I,i} = p_\alpha^{I,i} \circ l_s$ with $|J| = s$, and it is clear that the rank of l_s depends only on the rank $\rho(s)$ of the matrix $\partial P_\alpha^{I,i}/\partial u_J^\beta$ with $|I| = s$, $|J| = 2k-1-s$. Furthermore, the rank of leg_L may easily be seen to equal $\dim J^{k-1}\pi + \sum_{s=0}^{k-1} \rho(s)$, so that leg_L has maximal rank exactly when each component map l_s has maximal rank. In the special case of a one-dimensional base manifold M, we find that all the ranks $\rho(s)$ are equal, so that there is really only one condition to be satisfied, whereas in the more general case the ranks of the component maps may be different.

It is at this stage of the argument that we make use of the semi-holonomic jet manifold. If we examine the dependence of the functions $P^{I,i}_\alpha$ on the coordinates u^β_J with the given values of $|I|$ and $|J|$, we see that this dependence can only arise by taking $(k-1-s)$-fold total derivatives of the highest-order momentum functions $P^{K,i}_\alpha$ with $|K| = k-1$, and we have seen that these may be determined uniquely by extending the Lagrangian L to $\tau^*_k L : \widehat{J^k\pi} \longrightarrow \mathbb{R}$. The components of the matrix $\partial P^{I,i}_\alpha / \partial u^\beta_J$ are then just symmetrised combinations of the components of the Hessian matrix of L, and this is the form of the regularity condition given by Shadwick (1982).

The crucial consequence of this use of semi-holonomic jets is that the rank of the map leg_L no longer depends on any of the choices made in the construction of the Cartan form Θ_L, so that we may speak without ambiguity of a regular Lagrangian as one whose Legendre map leg_L has maximal rank. Indeed, we can even calculate what this maximal rank should be: it must be the same as the maximal rank of a Legendre map where all the local momentum functions $P^{I,i}_\alpha$ are totally symmetric. The number of such symmetric functions $P^{I+1_i}_\alpha$ with $0 \le |I| \le k-1$ equals $\dim J^k\pi - \dim E$; since the map leg_L is fibred over $J^{k-1}\pi$, it must take its values in a manifold of dimension no greater than $N(E,M,k) = \dim J^{k-1}\pi + \dim J^k\pi - \dim E$. Consideration of the example $L = \sum_{|I|=k}\sum_{\alpha=1}^n (u^\alpha_I)^2$ shows that this rank can actually be achieved. We see that the Legendre map leg_L for a regular Lagrangian is a submersion onto its image (although this is not a sufficient condition for regularity), and so leg_L will admit local sections. If, for a regular Lagrangian, there is a Legendre map leg_L which admits a global section Ψ, we shall say that L is *hyper-regular*; the composition $\mathrm{Leg}_L \circ \Psi$ will then be a global section of the restriction of the bundle $J^1\pi^\dagger_{k-1} \longrightarrow J^1\pi^*_{k-1}$ to the covariant phase space $\mathrm{Im}(\mathrm{leg}_L)$ and so will define a Hamiltonian system. Since the map leg_L is not injective, there will generally be many sections Ψ passing through any given point of $J^{2k-1}\pi$, and so the correspondence between the Euler–Lagrange structure and the Hamiltonian structure will in general not be unique.

6. The second-order case

According to the description of the Cartan form and Legendre map given section 4, the case of second-order field theories would seem to be subject to the same lack of uniqueness as any other higher-order case. There have, nevertheless, been several demonstrations that in second-order field theories a canonical choice of Cartan form may indeed be made, and we shall show in the present section that this is another consequence of the use of semi-holonomic jets. We shall also be able to give an explicit description of the covariant phase space. We shall not use multi-index notation for the second-order theory, but will instead let coordinates on $J^2\pi$ be $(x^i, u^\alpha, u^\alpha_i, u^\alpha_{ij})$ (with $u^\alpha_{ji} = u^\alpha_{ij}$) and those on $J^1\pi_1$ be $(x^i, u^\alpha, u^\alpha_i, u^\alpha_{,j}, u^\alpha_{i,j})$, with the appropriate modifications for the other manifolds which we shall need.

Given a map $\bar{L} : J^1\pi_1 \longrightarrow \mathbb{R}$ which extends a second-order Lagrangian $L : J^2\pi \longrightarrow \mathbb{R}$, the local momentum functions for $\bar{L}$ are given by

$$P = p \circ \mathrm{Leg}_{\bar{L}} = \bar{L} - u^\alpha_{,j}\frac{\partial \bar{L}}{\partial u^\alpha_{,j}} - u^\alpha_{i,j}\frac{\partial \bar{L}}{\partial u^\alpha_{i,j}}$$

$$P^{,j}_\alpha = p^{,j}_\alpha \circ \mathrm{Leg}_{\bar{L}} = \frac{\partial \bar{L}}{\partial u^\alpha_{,j}}$$

$$P_\alpha^{i,j} = p_\alpha^{i,j} \circ \mathrm{Leg}_{\bar{L}} = \frac{\partial \bar{L}}{\partial u^\alpha_{i,j}}$$

so that the Cartan form for $\bar{L}$ is

$$\Theta_{\bar{L}} = \left(\frac{\partial \bar{L}}{\partial u^\alpha_{,j}}(\mathrm{d}u^\alpha - u^\alpha_{,k}\mathrm{d}x^k) + \frac{\partial \bar{L}}{\partial u^\alpha_{i,j}}(\mathrm{d}u^\alpha_i - u^\alpha_{i,k}\mathrm{d}x^k) \right) \wedge \mathrm{d}^{m-1}x_j + \bar{L}\,\mathrm{d}^m x$$

and the Euler–Lagrange form is

$$\mathrm{E}(\mathrm{d}\bar{L}) = \left(\frac{\partial \bar{L}}{\partial u^\alpha} - \frac{\mathrm{d}}{\mathrm{d}x^j}\frac{\partial \bar{L}}{\partial u^\alpha_{,j}} \right) \mathrm{d}u^\alpha \wedge \mathrm{d}^m x + \left(\frac{\partial \bar{L}}{\partial u^\alpha_i} - \frac{\mathrm{d}}{\mathrm{d}x^j}\frac{\partial \bar{L}}{\partial u^\alpha_{i,j}} \right) \mathrm{d}u^\alpha_i \wedge \mathrm{d}^m x.$$

By writing the equation $\mathrm{d}L = \iota^*_{1,1}\mathrm{d}\bar{L}$ in coordinates and equating coefficients, we find that, on the holonomic manifold $J^2\pi$,

$$\begin{aligned}
\frac{\partial L}{\partial u^\alpha} &= \iota^*_{1,1}\frac{\partial \bar{L}}{\partial u^\alpha} \\
\frac{\partial L}{\partial u^\alpha_i} &= \iota^*_{1,1}\left(\frac{\partial \bar{L}}{\partial u^\alpha_i} + \frac{\partial \bar{L}}{\partial u^\alpha_{,i}} \right) \\
\frac{\partial L}{\partial u^\alpha_{ij}} &= \iota^*_{1,1}\frac{n(ij)}{2}\left(\frac{\partial \bar{L}}{\partial u^\alpha_{i,j}} + \frac{\partial \bar{L}}{\partial u^\alpha_{j,i}} \right)
\end{aligned}$$

where $n(ij)$ denotes the number of different indices represented by i and j, so that

$$\begin{aligned}
\iota^*_{1,1}\Theta_{\bar{L}} = \Bigg[&\left(\frac{\partial L}{\partial u^\alpha_j} - \iota^*_{1,1}\frac{\partial \bar{L}}{\partial u^\alpha_j} \right)(\mathrm{d}u^\alpha - u^\alpha_k \mathrm{d}x^k) \\
&+ \iota^*_{1,1}\frac{\partial \bar{L}}{\partial u^\alpha_{i,j}}(\mathrm{d}u^\alpha_i - u^\alpha_{ik}\mathrm{d}x^k) \Bigg] \wedge \mathrm{d}^{m-1}x_j + L\,\mathrm{d}^m x.
\end{aligned}$$

Now at each point $j^3_p\phi \in J^3\pi$, the pull-back $\iota^*_{2,1}\mathrm{E}(\mathrm{d}\bar{L})_{j^3_p\phi}$ is an $(m+1)$-covector on $J^1\pi$ which may be expressed as $\eta \wedge (\pi_1^*\Omega)_{j^1_p\phi}$, where $\eta \in T^*_{j^1_p\phi}J^1\pi$. The cotangent vector η is of course not unique, but the difference between any two such cotangent vectors is horizontal over M; it follows that $(S_\Omega)_{j^1_p\phi} \lrcorner\, \eta \in \bigwedge^m T^*_{j^1_p\phi}J^1\pi$ does not depend on the particular representative η. We shall let $S_\Omega(\mathrm{E}(\mathrm{d}\bar{L}))$ denote the m-form constructed by this method; it is an m-form on $J^3\pi$ semi-basic over E, and in coordinates may be written as

$$S_\Omega(\mathrm{E}(\mathrm{d}\bar{L})) = \iota^*_{2,1}\left(\frac{\partial \bar{L}}{\partial u^\alpha_j} - \frac{\mathrm{d}}{\mathrm{d}x^i}\frac{\partial \bar{L}}{\partial u^\alpha_{j,i}} \right) \mathrm{d}u^\alpha \wedge \mathrm{d}^{m-1}x_j.$$

If we set

$$\Theta_L = \pi^*_{2,1}\iota^*_{1,1}\Theta_{\bar{L}} + S_\Omega(\mathrm{E}(\mathrm{d}\bar{L}))$$

then in coordinates

$$\Theta_L = \left[\left(\frac{\partial L}{\partial u^\alpha_j} - \iota^*_{2,1} \frac{\mathrm{d}}{\mathrm{d}x^i} \frac{\partial \bar{L}}{\partial u^\alpha_{j,i}} \right) (\mathrm{d}u^\alpha - u^\alpha_k \mathrm{d}x^k) \right.$$
$$\left. + \iota^*_{2,1} \frac{\partial \bar{L}}{\partial u^\alpha_{i,j}} (\mathrm{d}u^\alpha_i - u^\alpha_{ik} \mathrm{d}x^k) \right] \wedge \mathrm{d}^{m-1} x_j + L\, \mathrm{d}^m x .$$

If we allow arbitrary extensions $\bar{L}$ of L then this expression is not defined uniquely, for the derivatives $\partial \bar{L}/\partial u^\alpha_{i,j}$ do not define unique functions on $J^3\pi$. We may, however, specify that any extension $\bar{L}$ should satisfy $\bar{L}|_{\widehat{J^2\pi}} = \tau_2^* L$, where the coordinate description of $\tau_2 : \widehat{J^2\pi} \longrightarrow J^2\pi$ is

$$u^\alpha_{ij} \circ \tau_2 = \frac{1}{n(ij)} \left(u^\alpha_{i,j} + u^\alpha_{j,i} \right) .$$

With this understanding, we find that

$$\frac{\partial \bar{L}}{\partial u^\alpha_{i,j}} = \frac{1}{n(ij)} \frac{\partial L}{\partial u^\alpha_{ij}}$$

and so we may make a canonical choice of Cartan form with coordinate representation

$$\Theta_L = \left[\left(\frac{\partial L}{\partial u^\alpha_j} - \frac{1}{n(ij)} \frac{\mathrm{d}}{\mathrm{d}x^i} \frac{\partial L}{\partial u^\alpha_{ij}} \right) (\mathrm{d}u^\alpha - u^\alpha_k \mathrm{d}x^k) \right.$$
$$\left. + \frac{1}{n(ij)} \frac{\partial L}{\partial u^\alpha_{ij}} (\mathrm{d}u^\alpha_i - u^\alpha_{ik} \mathrm{d}x^k) \right] \wedge \mathrm{d}^{m-1} x_j + L\, \mathrm{d}^m x .$$

Since the Legendre map $\mathrm{leg}_L : J^3\pi \longrightarrow J^1\pi_1^*$ corresponding to this canonical choice of Θ_L satisfies

$$p^{i,j}_\alpha \circ \mathrm{leg}_L = \frac{1}{n(ij)} \frac{\partial L}{\partial u^\alpha_{ij}} = p^{j,i}_\alpha \circ \mathrm{leg}_L$$

we may use this condition to determine the covariant phase space for L. Recall that the complement of $\iota_{1,1} J^2\pi$ in $\widehat{J^2\pi}$ is $\pi_1^* \bigwedge^2 T^*M \otimes \pi_{1,0}^* V\pi$, and that this may be regarded as an affine sub-bundle of $J^1\pi_1$ over $J^1\pi$. The fibre-affine maps $J^1\pi_1 \longrightarrow \mathbb{R}$ which are constant on the fibres of this sub-bundle define a sub-bundle $(\pi_1^* \bigwedge^2 T^*M \otimes \pi_{1,0}^* V\pi)^c$ of $J^1\pi_1^\dagger$, and this construction passes to the quotient to give a sub-bundle $(\pi_1^* \bigwedge^2 T^*M \otimes \pi_{1,0}^* V\pi)^\circ$ of $J^1\pi_1^*$; the notation is chosen because this vector bundle is canonically isomorphic to the annihilator of $\pi_1^* \bigwedge^2 T^*M \otimes \pi_{1,0}^* V\pi$ in $(\pi_1^* T^*M \otimes V\pi_1)^*$, the dual of the vector bundle associated to $J^1\pi_1$. It follows from this description that the total space of $(\pi_1^* \bigwedge^2 T^*M \otimes \pi_{1,0}^* V\pi)^\circ$ is the submanifold of $J^1\pi_1^*$ defined locally by $p^{i,j}_\alpha = p^{j,i}_\alpha$, so we see that leg_L takes its values in this manifold. Furthermore, the dimension of this manifold is $(m + n + mn) + (mn + m^2 n - \frac{1}{2} mn(m-1))$, which equals $N(E, M, 2) = \dim J^2\pi + \dim J^1\pi - \dim E$; it follows that $(\pi_1^* \bigwedge^2 T^*M \otimes \pi_{1,0}^* V\pi)^\circ$ may be regarded as the covariant phase space for second-order Lagrangians.

Acknowledgments

The research reported on in this paper is part of a programme which is supported by NATO, under the Collaborative Research Grants Programme, and by the Belgian National Fund for Scientific Research. The authors wish to thank the staff of the Instituut voor Theoretische Mechanika, Rijksuniversiteit Gent, Belgium, for their hospitality while part of this work was carried out; and they are grateful to Willy Sarlet, Frans Cantrijn and Geoff Prince for their comments and their forbearance.

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